A PSEUDO-DIFFERENTIAL CALCULUS ON GRADED NILPOTENT LIE GROUPS

VÉRONIQUE FISCHER AND MICHAEL RUZHANSKY

Abstract. In this paper, we present first results of our investigation regarding symbolic pseudo-differential calculi on nilpotent Lie groups. On any graded Lie group, we define classes of symbols using difference operators. The operators are obtained from these symbols via the natural quantisation given by the representation theory. They form an algebra of operators which shares many properties with the usual Hörmander calculus.

1. Introduction

In the last five decades pseudo-differential operators have become a standard tool in the study of partial differential equations. It is natural to try to define analogues of the Euclidean pseudo-differential calculus in other settings. On one hand, while it is always possible to obtain a local calculus on any (connected) manifold, the question becomes much harder for global calculi. If, in addition, one requires a notion of symbol, the quasi inherent context is the one of Lie groups of type I where a Plancherel-Fourier analysis is available. On the other hand, from the viewpoint of what can actually be done at the level of operators, the investigation should start in the context of Lie groups with polynomial volume-growth where analysis of integral operators is quite well understood. Therefore the natural setting to start developing global pseudo-differential calculi is nilpotent or compact Lie groups, together with their semi-direct products.

The genesis of this idea began quite some time ago; if a starting line had to be drawn, it would be in the seventies with the work of Elias Stein (and his collaborators Folland, Rotschild, etc.), and continued, in the next decade, with the work of Beals and Greiner amongst many others. Their motivation came from the study of differential operators on CR or contact manifolds, modelling locally the operators on homogeneous left-invariant convolution operators on nilpotent groups (cf [13]). In 'practice' and from this motivation, only nilpotent Lie groups endowed with some compatible structure of dilations, i.e. graded Lie groups, are considered. This is also the setting of our present investigation.

Since the seventies, several global calculi of operators on homogeneous Lie groups have appeared. However they were often calculi of left-invariant operators with the

Date: July, 2012.

1991 *Mathematics Subject Classification.* Primary 35S05; Secondary 43A80.

Key words and phrases. Pseudo-differential operators, nilpotent Lie groups.

The first author acknowledges the support of the London Mathematical Society via the Grace Chisholm Fellowship. It was during this fellowship held at King's College London in 2011 that the work was initiated. The second author was supported in part by the EPSRC Leadership Fellowship EP/G007233/1.

following notable exceptions to the authors' knowledge. Beside Dynin's construction of certain operators on the Heisenberg group in [6], a non-invariant pseudo-differential calculus on any homogeneous group was developed in [3] but this is not symbolic since the operator classes are defined via properties of the kernel. In the revised version of [17], Taylor describes several (non-invariant) operator calculi and, in a different direction, he also explains a way to develop symbolic calculi: using the representations of the group, he defines a general *quantisation* and the natural *symbols* on any unimodular type I group (by quantisation, we mean a procedure which associates an operator to a symbol). He illustrates this on the Heisenberg group and obtains there several important applications for, e.g., the study of hypoellipticity. He uses that fact that, because of the properties of the Schrödinger representations of the Heisenberg group, a symbol is a family of operators in the Euclidean space, themselves given by symbols via the Weyl quantisation. Recently, the definition of suitable classes of Shubin type for these Weyl-symbols led to another version of the calculus on the Heisenberg group in [1].

Recently as well, using the global quantisation procedure noted in [17], the second author and Turunen developed a global symbolic calculus on any compact Lie group in [14]. They successfully defined symbol classes so that the quantisation procedure makes sense and the resulting operators form an algebra of operators with properties 'close enough' to the one enjoyed by the Euclidean Hörmander calculus (in fact, in a later work with Wirth [16], they showed that the calculus in [14] leads to the usual Hörmander operator classes on $\mathbb{R}^n$ extended to compact connected manifolds). Their approach is valid for any compact Lie group whereas the calculus of [1] is very specific to the Heisenberg group. The crucial and new ingredient in the definition of symbol classes in [14] was defining *difference operators* in order to replace the Euclidean derivatives in the Fourier variables. These difference operators allow expressing the pseudo-differential behaviour directly on the group.

In our present investigations, we build upon this notion to study operators in the nilpotent setting. However it is not possible to extend readily the results of the compact case developed in [14] to the nilpotent context. Some technical difficulties appear because, for example, the dual of G is no longer discrete and the unitary irreducible representations are almost all infinite dimensional. More problematically there is no Laplace-Beltrami operator and one expects to replace it by a sub-Laplacian on stratified Lie groups or, more generally, by a positive Rockland operator $\mathcal{L}$ on graded Lie groups; such operators are not central. Hence new technical ideas are needed to develop a pseudo-differential calculi on graded Lie groups using the natural quantisation together with the notion of difference operators from [14].

The results that we have obtained in our investigation of this question so far were presented in the talk given by the first author at the conference *Fourier analysis and pseudo-differential operators*, Aalto University, 25-30 June, 2012. They are the following (here $1 \geq \rho \geq \delta \geq 0$):

- **(R1):** The symbol classes form an algebra of operators $\cup_{m\in\mathbb{R}} S^m_{\rho,\delta}$ stable by taking the adjoint.
- **(R2):** Let $\rho \neq 0$. The operators obtained by quantisation from $\cup_{m\in\mathbb{R}} S^m_{\rho,\delta}$ form an algebra of operators $\cup_{m\in\mathbb{R}} \Psi^m_{\rho,\delta}$ stable by taking the adjoint.
- **(R3):** The set of operators $\cup_{m\in\mathbb{R}} \Psi^m_{\rho,\delta}$ contains the left-invariant calculus.

(R4): The kernels are of Calderon-Zygmund type on homogeneous Lie groups; in particular our operators of order 0 are more singular than their Euclidean counterparts.

(R5): If $\rho \in [0,1)$, then the operators in $\Psi^0_{\rho,\rho}$ are continuous on $L^2(G)$.

(R6): $(1+\mathcal{L})^{\frac{m}{\nu}} \in \Psi^m_{1,0}$, where $\mathcal{L}$ is a positive Rockland operator of degree ν, see Section 2.2.

(R7): Positive operators of the calculus satisfy sharp Gårding inequalities.

As a consequence from Results (R2), (R5) and (R6), if $\rho \neq 0$, any pseudo-differential operator is continuous on the Sobolev spaces with the loss of derivatives being controlled by the order. All those properties justify, from our viewpoint, the choice of vocabulary of pseudo-differential calculi.

In this paper, due to the lack of space, we will state and prove the following parts of Results (1-7). Result (R1) is proved in Subsection 4.1. The proof of Result (R2) is given in Subsection 4.4 but relies on Result (R4) which is stated in Subsection 4.2 and only partially proved (although a weaker version of (R4) is obtained and would suffice to prove (R2) completely but lengthily). In Subsection 3.5, Result (R3) is stated and proved while (R6) is stated in greater generality but not proved in this paper. The precise statements and proofs of Results (R5) and (R7) will appear elsewhere.

The paper is organised as follows. In Section 2, we explain the precise setting of our investigation for the group, the Sobolev spaces involved here and the group Fourier transform. In Section 3, we precise our definition of quantisation and symbol classes. In Section 4, we give the properties of the symbols and of the corresponding kernels and operators stated above.

Convention: All along the paper, C denotes a constant which may vary from line to line. We denote by $\lceil r \rceil$ the smallest integer which is strictly greater than the real r.

2. Preliminaries

In this section, we set some notation and recall some known properties regarding the groups under investigation, the Taylor expansion in this context and representation theory.

2.1. **The group G.** Here we recall briefly the definition of graded nilpotent Lie groups and their natural homogeneous structure. A complete description of the notions of graded and homogeneous nilpotent Lie group may be found in [8, Chapter 1].

Let G be a graded Lie group. This means that G is a connected simply connect nilpotent Lie group whose Lie algebra $\mathfrak{g}$ may be decomposed as the sum of subspaces,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s \quad \text{such that} \quad [\mathfrak{g}_{j_1}, \mathfrak{g}_{j_2}] \subset \begin{cases} \mathfrak{g}_{j_1+j_2} & \text{if } j_1 + j_2 \leq s \ , \\ 0 & \text{if } j_1 + j_2 > s \ . \end{cases}$$

Examples of such groups are the Heisenberg group and more generally any stratified groups (which by definition correspond to the case $\mathfrak{g}_1$ generating the full Lie algebra $\mathfrak{g}$).

Since the exponential mapping is a diffeomorphism from $\mathfrak{g}$ to G, we can identify G with $\mathbb{R}^n$ or $\mathfrak{g}_1 \times \ldots \times \mathfrak{g}_s$ (as a manifold). This means that we identify a point

$(x_1, \ldots, x_s)$ in $\mathfrak{g}_1 \times \ldots \times \mathfrak{g}_s$ with the point $x = \exp(x_1 + \ldots + x_s)$ of the group G. Consequently we allow ourselves to denote by $C(G)$, $C_c^\infty(G)$ and $\mathcal{S}(G)$ etc... the spaces of continuous functions, of smooth and compactly supported functions or of Schwartz functions on G identified with $\mathbb{R}^n$.

The group G is endowed with a natural homogeneous structure, namely, we define the *dilations* on G by

$$r \cdot (x_1, x_2, \ldots, x_s) = (rx_1, r^2x_2, \ldots, r^sx_s) \quad , \quad r > 0 \ , \ (x_1, x_2, \ldots, x_s) \in G \ .$$

Recall that the mappings $e^t : g \in G \mapsto e^t \cdot g$, $t \in \mathbb{R}$, form a one parameter group $\{e^t \ , \ t \in \mathbb{R}\}$ of automorphisms of G. The natural notions of homogeneous functions, distributions and operators follow.

Recall that a *homogeneous norm* on G is a continuous function $|\cdot| : G \to [0, +\infty)$ homogeneous of degree 1 on G which vanishes only at 0. Any homogeneous norm satisfies a triangular inequality up to a constant. Any two homogeneous norms are equivalent.

Various aspects of analysis on G can be developed in a comparable way with the Euclidean setting [4], sometimes replacing the topological dimension

$$n := \dim G = n_1 + \ldots + n_s \quad , \quad \text{with } n_j := \dim \mathfrak{g}_j \ ,$$

of the group G by its homogeneous dimension

$$Q := n_1 + 2n_2 + \ldots + sn_s \ .$$

Throughout this paper, we will adopt the shorthand notation $\mathbb{N}^n$ for $\mathbb{N}^{n_1} \times \ldots \times \mathbb{N}^{n_s}$, where we agree that $\mathbb{N}$ also contains 0. Hence by $\alpha \in \mathbb{N}^n$ we mean the multi-index formed by the multi-indices $\alpha_j \in \mathbb{N}^j$, $j = 1, \ldots, s$, i.e. $\alpha = (\alpha_1, \ldots, \alpha_s)$.

2.1.1. *The polynomial functions q_α.* For each $j = 1, \ldots, s$, we fix a basis $X_{j,i}$, $1 \le i \le n_j$ of $\mathfrak{g}_j$. For $\beta = (\beta_1, \ldots, \beta_{n_j}) \in \mathbb{N}^{n_j}$, x^β means the polynomial on $\mathfrak{g}_j$ given by

$$x_1 X_{j,1} + \ldots + x_{n_j} X_{j,n_j} \longmapsto x_1^{\beta_1} \ldots x_{n_j}^{\beta_{n_j}} \ ;$$

we set $|\beta| = \beta_1 + \ldots + \beta_{n_j}$ and $\beta! = \beta_1! \ldots \beta_{n_j}!$. Now for $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^n$ we set

$$x^\alpha := x_1^{\alpha_1} \ldots x_s^{\alpha_s} \quad , \quad q_\alpha(x) := \frac{1}{\alpha!} x^\alpha \quad , \quad \text{where } \alpha! = \alpha_1! \ldots \alpha_s! \ .$$

The function q_α is homogeneous of degree

$$[\alpha] := \alpha_1 + 2\alpha_2 + \ldots + s\alpha_s \ .$$

The Baker-Campbell-Hausdorff formula and the homogeneity imply

$$q_\alpha(xy) = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2} q_{\alpha_1}(x) q_{\alpha_2}(y) \ , \tag{2.1}$$

where the coefficients c_{α_1,α_2} are real and, moreover,

$$c_{\alpha_1,0} = \begin{cases} 1 & \text{if } \alpha_1 = \alpha \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad c_{0,\alpha_2} = \begin{cases} 1 & \text{if } \alpha_2 = \alpha \\ 0 & \text{otherwise} \end{cases} \ .$$

For any positive integer p divisible by $2, \ldots, s$, the function

$$\left(\sum_{[\alpha]=p} |q_\alpha|^2 \right)^{1/(2p)} \tag{2.2}$$

is a homogeneous norm.

2.1.2. *The invariant vector fields.* Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping Lie algebra of $\mathfrak{g}$. It is formed by all the finite linear combinations of X^α, $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^n$, where we have set:

$$\text{first } X^{\alpha_j} := X_{j,1}^{\alpha_{j,1}} \ldots X_{j,n_j}^{\alpha_{j,n_j}} \quad \text{and then} \quad X^\alpha := X^{\alpha_1} \ldots X^{\alpha_s} \ .$$

If X is a vector of the Lie algebra $\mathfrak{g}$, we keep the same notation for the corresponding left-invariant vector field and we denote by $\tilde{X}$ the corresponding right-invariant vector field. This means that

$$Xf(x) = \partial_{t=0} f(xe^{tX}) \quad , \quad \tilde{X}f(x) = \partial_{t=0} f(e^{tX}x) \ .$$

It yields the natural correspondence between $\mathfrak{U}(\mathfrak{g})$ and the differential operators which are invariant under either left or right translations; this means that any differential operator on G which is invariant under either left or right translation can be written as a finite linear combination of X^α or $\tilde{X}^\alpha$, respectively. Note that the differential operators X^α and $\tilde{X}^\alpha$ are homogeneous of degree $[\alpha]$ and that any left or right-invariant differential operator which is homogeneous of degree d can be written as the finite linear combination over $[\alpha] = d$:

We will need the following property:

Lemma 2.1. *For any $\alpha \in \mathbb{N}^n$, there exists a constant $C = C_\alpha > 0$ such that for any $f \in C^{[\alpha]}(G)$ and any $y \in G$,*

$$|X_x^\alpha \{f(xy)\}| \leq C(1 + \max_{i,j} |y_{j,i}|)^{(s-1)|\alpha|} \max_{[\beta] \leq s[\alpha]} \left| X^\beta f(xy) \right| \ .$$

Proof of Lemma 2.1. We will use the adjoint representations Ad and ad of the group G and the algebra $\mathfrak{g}$ on $\mathfrak{U}(\mathfrak{g})$. Because of the exponential properties,

$$\mathrm{Ad}(y)X = X + \sum_{j=1}^{s-1} \frac{1}{j!} \mathrm{ad}(\ln y)^j(X) = \sum_{|\gamma| \leq s-1} c_\gamma y^\gamma \prod_{l,k} \mathrm{ad}(X_{l,k})^{\gamma_{l,k}} X \ ,$$

with the ordering in the product decided earlier. This remark together with the formula

$$X_x^\alpha \{f(xy)\} = \left(\prod_{j,i} \mathrm{Ad}(y^{-1}) X_{j,i}^{\alpha_{j,i}} \ f \right)(xy) \ ,$$

which comes from

$$X_x\{f(xy)\} = \partial_{t=0} f(xyy^{-1}e^{tX}y) = \left(\mathrm{Ad}(y^{-1})X \ f\right)(xy) \ ,$$

yields the result. □

2.1.3. *Taylor expansions on G.* In the setting of graded Lie groups one can obtain the left or right mean value theorem and left or right Taylor expansions adapted to the homogeneous structure [8, Theorem 1.42]. Let us give the statement for left invariance. We will need the following definition: the (left) Taylor polynomial of homogeneous degree M of a function $f \in C^{M+1}(G)$ at a point $x \in G$ is by definition the polynomial $P_{x,M}^{(f)}$ satisfying

$$X^\alpha P_{x,M}^{(f)}(0) = \begin{cases} X^\alpha f(x) & \text{whenever} \quad \alpha \in \mathbb{N}^n \text{ with } [\alpha] \leq M \ , \\ 0 & \text{if } [\alpha] > M \ . \end{cases}$$

We also define the remainder to be

$$R^{(f)}_{x,M}(z) := f(xz) - P^{(f)}_{x,M}(z) \ .$$

From the following easy computations,

$$X^\alpha \tilde{X}^{\tilde{\alpha}} q_\beta(0) = \left\{ \begin{array}{ll} 0 & \text{if } \alpha+\tilde{\alpha}\neq\beta \\ 1 & \text{if } \alpha+\tilde{\alpha}=\beta \end{array} \right. ,$$

and, implicitly, all the choices made in the writing of the polynomials q_α and of the operators X^α, we can write the Taylor polynomial as

$$P^{(f)}_{x,M} = \sum_{[\alpha]\leq M} X^\alpha f(x) q_\alpha \ .$$

Proposition 2.2 (Mean value and Taylor expansion [8])**.** *Let us fix a homogeneous norm $|\cdot|$ on G.*

(1) *(Mean value property) There exist positive group constants C_0 and b such that for any function $f\in C^1(G)$, we have*

$$|f(xy)-f(x)| \leq C_0 \sum_{j=1}^{s} |y|^j \sup_{\substack{|z|\leq b|y| \\ 1\leq i\leq n_j}} |X_{ij} f(xz)|$$

(2) *(Taylor expansion) For each $M\in\mathbb{N}$ there exist positive group constants C_M such that for any function $f\in C^{M+1}(G)$, we have*

$$\forall y\in G \qquad |R^{(f)}_{x,M}(y)| \leq C_M \sum_{\alpha} |y|^{[\alpha]} \sup_{|z|\leq b^{M+1}|y|} |X^\alpha f(xz)| \ ,$$

where the sum is over α in

$$S_M := \{\alpha\in\mathbb{N}^n \ : \ [\alpha]>M \ , \ |\alpha|\leq j_o + \max_{[\beta]\leq M}|\beta|\} \ . \tag{2.3}$$

where j_o is the smallest $j\in\mathbb{N}$ such that $\mathfrak{g}_j$ is non-trivial.

The control can be improved in the stratified case (again see [8]) but we present here the more general case of the graded groups.

Remark 2.3. Proposition 2.2 extends easily to functions which are vector valued in a Banach space, replacing the modulus by operator norms.

2.2. **A positive Rockland operator $\mathcal{L}$.** We choose $\mathcal{L}$ a positive (left) Rockland operator of homogeneous degree ν. Let us recall that being a Rockland operator means that $\mathcal{L}$ is a differential operator on G which is left-invariant and homogeneous of degree ν and such that for every non-trivial irreducible representation π of G, the operator $\pi(\mathcal{L})$ is injective on smooth vectors (see Section 2.3 for the definition of $\pi(\mathcal{L})$.); being positive means

$$\forall f\in\mathcal{S}(G) \qquad (\mathcal{L}f,f)_{L^2(G)}\geq 0 \ .$$

Here as usual

$$(f_1,f_2)_{L^2(G)} = \int_G f_1(g)\overline{f_2(g)}dg \ ,$$

and dg denotes the Haar measure $dg = dx_{1,1}\ldots dx_{s,s}$.

In the stratified case, we choose $\mathcal{L}$ to be the sub-Laplacian $-\sum_{i=1}^{n_1} X_{1,i}^2$ (and so $\nu = 2$). In the graded case, if ν_o denotes some common multiple of $1, 2, \ldots, s$ then

$$\sum_{1\leq j\leq s} \sum_{1\leq i\leq n_j} (-1)^{\frac{\nu_o}{j}} X_{j,i}^{2\frac{\nu_o}{j}} \quad \text{and} \quad \sum_{1\leq j\leq s} \sum_{1\leq i\leq n_j} X_{j,i}^{4\frac{\nu_o}{j}}$$

are positive Rockland operators of degree $2\nu_o$ and $4\nu_o$ respectively. Hence we can always find an operator $\mathcal{L}$ (Rockland or sub-Laplacian) of degree $\nu < Q$ on any graded nilpotent Lie group G.

By the celebrated result of Helffer and Nourrigat [10], $\mathcal{L}$ is hypoelliptic and satisfies subelliptic estimates. Furthermore [8, ch. 4.B] it admits an essentially self-adjoint extension on $C_c^\infty(G)$ for which we keep the same notation $\mathcal{L}$. Let E denote the spectral measure of $\mathcal{L}$. For any measurable function ϕ on $[0,\infty)$, we define the operator

$$\phi(\mathcal{L}) := \int_0^\infty \phi(\lambda) dE_\lambda \ ,$$

which is invariant under left-translation; if it maps continuously $\mathcal{S}(G) \to \mathcal{S}'(G)$ (for example if ϕ is bounded), by the Schwartz kernel theorem, it is a convolution operator with kernel $\phi(\mathcal{L})\delta_o \in \mathcal{S}'(G)$, that is,

$$\phi(\mathcal{L})f = f \ * \ \phi(\mathcal{L})\delta_o \quad , \quad f \in \mathcal{S}(G) \ ;$$

recall that the group convolution is defined via

$$f_1 * f_2(g) = \int_G f_1(g') f_2({g'}^{-1} g) dg' \quad , \quad f_1, f_2 \in \mathcal{S}(G) \ .$$

The above hypotheses ensure the following Marcinkiewicz-type properties proved by A. Hulanicki [11].

Proposition 2.4 (Hulanicki)**.** *For any $\alpha, \beta \in \mathbb{N}^n$, there exists $k = k_{\alpha,\beta} \in \mathbb{N}$ and $C = C_{\alpha,\beta} > 0$ such that for any $\phi \in C^\infty([0,\infty))$ we have*

$$\|x^\alpha X^\beta \phi(\mathcal{L})\delta_o\|_{L^1(G)} \leq C \sup_{\lambda\geq 0,\, k_1\leq k} (1+\lambda)^k \partial_\lambda^{k_1} |\phi(\lambda)| \ ;$$

by this we mean that if the supremum in the right hand side is finite, then the distribution $x^\alpha X^\beta \phi(\mathcal{L})\delta_o$ coincides with an integrable function and the inequality holds.

Remark 2.5. Consequently if $\phi \in \mathcal{S}(\mathbb{R})$ is Schwartz, then the kernel $\phi(\mathcal{L})\delta_o$ is also Schwartz on G, i.e. $\phi(\mathcal{L})\delta_o \in \mathcal{S}(G)$.

2.2.1. *Symmetric operators and functions.* We say that an operator $T : \mathcal{S}(G) \to \mathcal{S}'(G)$ is *symmetric* when it satisfies $T^t = T$, that is,

$$\int T f_1(g) \ f_2(g) dg = \int f_1(g) \ T f_2(g) dg \ .$$

A function f on G is *symmetric* whenever $f^t = f$ where

$$f^t(x) = f(x^{-1}) \ .$$

It is easy to see that a positive Rockland operator $\mathcal{L}$ is symmetric if and only if $\mathcal{L}$ is a linear combination of X^α with $\alpha \in \mathbb{N}^n$ of even isotropic degree, i.e. $|\alpha| \in 2\mathbb{N}$. The operators $\mathcal{L}$ given as examples above are symmetric.

If $\mathcal{L}$ is symmetric, the convolution kernels $\phi(\mathcal{L})\delta_o$ of multipliers in $\mathcal{L}$ are symmetric.

We denote by $\tilde{\mathcal{L}}$ the right-invariant differential operator which corresponds to the same element of $\mathfrak{U}(\mathfrak{g})$ as $\mathcal{L}$. One checks easily that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ commute; they also commute strongly, i.e. their spectral measures commute. If $\mathcal{L}$ is symmetric, then for any function $f \in \mathcal{S}(G)$: $\tilde{\mathcal{L}}(f^t) = (\mathcal{L}f)^t$.

2.2.2. *The heat kernel h.* The $\mathcal{L}$*-heat kernel* h_t is defined as the kernel of $\exp(-t\mathcal{L})$ for each $t > 0$. It is easy to show the following homogeneity property,

$$h_t(x) = t^{-\frac{Q}{\nu}} h(t^{-\frac{1}{\nu}} x) \quad \text{with} \quad h = h_1 \in \mathcal{S}(G) \ .$$

The heat kernel h is Schwartz.

2.2.3. *Bessel potential.* The definition of Bessel potentials is well-known and used in the stratified case for the sub-Laplacian [7]. It is not difficult to see that the following properties still hold in the graded case.

Let $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$. The integral

$$G_a(x) = \frac{1}{\Gamma(\frac{a}{\nu})} \int_0^\infty t^{\frac{a}{\nu}-1} e^{-t} h_t(x) dt \ ,$$

converges absolutely for $x \neq 0$ and defines the Bessel potential $G_a \in C^\infty(G\backslash\{0\})$. The Bessel potential is an integrable function,

$$\|G_a\|_{L^1(G)} \leq \frac{\Gamma(\operatorname{Re} \frac{a}{\nu})}{|\Gamma(\frac{a}{\nu})|} \|h\|_{L^1} < \infty \quad , \quad a \in \mathbb{C} \ , \ \operatorname{Re} a > 0 \ .$$

Using the properties of semigroup of $e^{-t\mathcal{L}}$, one obtains that $\|G_a\|_{L^2(G)}$ is square integrable if $\operatorname{Re} a > Q/2$.

The Bessel potential is the convolution kernel of the $L^2(G)$-bounded left-invariant operator $(I+\mathcal{L})^{-a/\nu}$ and of the $L^2(G)$-bounded right-invariant operator $(I+\tilde{\mathcal{L}})^{-a/\nu}$, so that we have

$$(I + \mathcal{L})^{-a/\nu} f = f * G_a \quad , \quad (I + \tilde{\mathcal{L}})^{-a/\nu} f = G_a * f \quad , \quad f \in L^2(G) \ .$$

2.2.4. *Sobolev spaces.* For $a \geq 0$, we define the $\mathcal{L}$-Sobolev spaces as the domain of $(I + \mathcal{L})^{\frac{a}{\nu}}$, that is,

$$L^2_a(G) = \{f \in L^2(G) \ , \ (I + \mathcal{L})^{\frac{a}{\nu}} f \in L^2(G)\} \ .$$

For $a < 0$, $L^2_a(G)$ is the completion of $L^2(G)$ for the pre-norm $f \mapsto \|(I+\mathcal{L})^{\frac{a}{\nu}} f\|_{L^2(G)}$. It is easy to see that for any $a \in \mathbb{R}$, the Sobolev space $L^2_a(G)$ is a Hilbert space for the norm

$$\|f\|_{L^2_a(G)} := \|(I + \mathcal{L})^{\frac{a}{\nu}} f\|_{L^2(G)} \ .$$

Adapting the stratified case [7], one obtains:

Proposition 2.6 (Sobolev spaces)**.** *Let $\mathcal{L}$ be a symmetric positive Rockland operator of homogeneous degree $\nu < Q$.*

(1) *If $a \leq b$, then $\mathcal{S}(G) \subset L^2_b(G) \subset L^2_a(G) \subset \mathcal{S}'(G)$ and an equivalent norm for $L^2_b(G)$ is $f \mapsto \|f\|_{L^2_a(G)} + \|\mathcal{L}^{\frac{b-a}{\nu}} f\|_{L^2_a(G)}$.*
(2) *If $a \in \mathbb{N}$, then an equivalent norm is given by $f \mapsto \sum_{[\alpha]\leq a} \|X^\alpha f\|_{L^2(G)}$.*
(3) *The dual space of $L^2_a(G)$ is isomorphic to $L^2_{-a}(G)$ via the bilinear form $(f_1, f_2) \mapsto \int_G f_1 f_2 dg$.*

(4) *We have the usual property of interpolation for Sobolev spaces: let T be a linear mapping from $L^2_{a_0}(G)+L^2_{a_1}(G)$ to locally integrable functions on G; we assume that T maps $L^2_{a_0}(G)$ and $L^2_{a_1}(G)$ boundedly into $L^2_{b_0}(G)$ and $L^2_{b_1}(G)$, respectively. Then T extends uniquely to a bounded mapping from $L^2_{a_t}(G)$ to $L^2_{b_t}(G)$ with $(a_t,b_t)=t(a_0,b_0)+(1-t)(a_1,b_1)$.*

Consequently, the Sobolev spaces do not depend on the choice of operators $\mathcal{L}$ as in the statement above. Such operators exist and we fix one of them until the end of the paper.

From the properties of the Bessel potential, the Sobolev inequalities are now easy to obtain.

Lemma 2.7 (Sobolev inequality)**.** *If $a>Q/2$ then any function $f\in L^2_a(G)$ admits a continuous bounded representative which satisfies*

$$\|f\|_{L^\infty(G)}\leq C_a\|f\|_{L^2_a(G)}\ ,$$

with $C_a=\|G_a\|_{L^2(G)}$ independent of f.

Sketch of the proof. It suffices to write

$$f=(I+\mathcal{L})^{-\frac{a}{\nu}}(I+\mathcal{L})^{\frac{a}{\nu}}f=\{(I+\mathcal{L})^{\frac{a}{\nu}}f\}*G_a\ .$$

□

2.3. **The unitary dual and the group Fourier transform.** We denote by $\widehat{G}$ the unitary dual of the group G, that is, the set of (strongly continuous) unitary irreducible representations modulo unitary equivalence. We will often identify a unitary irreducible representation π of G and its equivalence class; we denote the representation Hilbert space by $\mathcal{H}_\pi$ and the subspace of smooth vectors by $\mathcal{H}_\pi^\infty$.

The group Fourier transform of a function $f\in L^1(G)$ at $\pi\in\widehat{G}$ is the bounded operator $\widehat{f}(\pi)$ (sometimes this will be also denoted by $\pi(f)$ for longer expressions) on $\mathcal{H}_\pi$ given by

$$(\widehat{f}(\pi)v_1,v_2)_{\mathcal{H}_\pi}:=\int_G f(g)(\pi(g)^*v_1,v_2)_{\mathcal{H}_\pi}dg\quad,\quad v_1,v_2\in\mathcal{H}_\pi\ .$$

One can readily see the equality $\widehat{f_1*f_2}(\pi)=\widehat{f}_2(\pi)\widehat{f}_1(\pi)$.

The group Fourier transform of a vector $X\in\mathfrak{g}$ at $\pi\in\widehat{G}$ is the operator $\pi(X)$ on $\mathcal{H}_\pi^\infty$ given by

$$(\pi(X)v_1,v_2)_{\mathcal{H}_\pi}:=\partial_{s=0}(\pi(e^{sX})v_1,v_2)_{\mathcal{H}_\pi}\quad,\quad v_1,v_2\in\mathcal{H}_\pi^\infty\ .$$

Setting $\pi(X^\alpha)=\pi(X)^\alpha$, this yields the definition of the group Fourier transform of any element of $\mathfrak{U}(\mathfrak{g})$. With this notation we have for any $\alpha\in\mathbb{N}^n$,

$$\widehat{X^\alpha f}(\pi)=\pi(X)^\alpha\widehat{f}(\pi)=\pi(X^\alpha)\widehat{f}(\pi)\text{ and }\widehat{\tilde{X}^\alpha f}(\pi)=\widehat{f}(\pi)\pi(X)^\alpha=\widehat{f}(\pi)\pi(X^\alpha)\,,$$

with the convention for $\alpha=0$ that $\pi(X^0)=\pi(I)=I=\pi(X)^0$.

The properties above help in the systematic computations of certain expressions; for example we see $\pi(\{Xf_1\}*f_2)=\pi(f_2)\pi(X)\pi(f_1)=\pi(\tilde{X}f_2)\pi(f_1)$ and this is coherent with the direct and more tedious computation $\{Xf_1\}*f_2=f_1*\{\tilde{X}f_2\}$.

We may allow ourselves to write $\pi\mathcal{L}=\pi(\mathcal{L})$ for convenience.

2.4. **The Plancherel Theorem and the algebra $C^*(G)$.** About representation theory and the Plancherel theorem, we refer the reader to Dixmier's standard textbook [5].

Recall that a bounded operator A on a Hilbert space $\mathcal{H}$ is in the Hilbert-Schmidt class whenever $\|A\|_{HS}=\sqrt{\mathrm{Tr}\,(A^*A)}$ is finite. If $f\in L^2(G)\cap L^1(G)$ then $\widehat{f}(\pi)$ is a Hilbert-Schmidt operator, and the *Plancherel formula* holds,

$$\int_G |f(g)|^2 dg = \int_{\widehat{G}} \|\widehat{f}(\pi)\|_{HS}^2 d\mu(\pi) \ ,$$

where μ is the Plancherel measure on $\widehat{G}$. The group Fourier transform extends unitarily to $L^2(G)$ and a square integrable function $f\in L^2(G)$ gives rise to a μ-square-integrable field of Hilbert-Schmidt operators $\{\widehat{f}(\pi)\}$. Conversely, a μ-square-integrable field of Hilbert-Schmidt operators $\{\sigma_\pi\}$ defines a square integrable function f with

$$(f,f_1)_{L^2(G)} = \int_{\widehat{G}} \mathrm{Tr}\left(\sigma_\pi \ \widehat{f}_1(\pi)^*\right) d\mu(\pi) \quad , \quad f_1\in L^2(G) \ .$$

If $T:L^2(G)\to L^2(G)$ is a bounded linear operator which commutes with left translations, then there exists a μ-measurable field of uniformly bounded operators $\{\sigma_\pi^{(T)}\}$ such that for any $f\in L^2(G)$ the Hilbert-Schmidt operators $\widehat{Tf}(\pi)$ and $\sigma_\pi^{(T)}\widehat{f}(\pi)$ are equal μ-almost everywhere; the field $\{\sigma_\pi^{(T)}\}$ is unique up to a μ-negligible set. Note that by the Schwartz kernel theorem, the operator T is of convolution type with kernel $\kappa\in\mathcal{S}'(G)$,

$$Tf = f*\kappa \quad , \quad f\in\mathcal{S}(G) \ .$$

Conversely given a μ-measurable field of uniformly bounded operators $\{\sigma_\pi\}$ there exists a unique bounded linear operator $T:L^2(G)\to L^2(G)$ which commutes with translations and satisfies $\widehat{Tf}(\pi)=\sigma_\pi\widehat{f}(\pi)$ μ-almost everywhere.

If $\kappa\in\mathcal{S}'(G)$ is such that the corresponding convolution operator $f\in\mathcal{S}(G)\mapsto f*\kappa$ extends to a bounded operator $T:L^2(G)\to L^2(G)$ then we abuse the notation by setting $\sigma_\pi^{(T)}:=\pi(\kappa)\equiv\widehat{\kappa}(\pi)$. We denote by $C^*(G)$ the space of such distributions κ. Endowed with the (essential) supremum norm on $\widehat{G}$,

$$\|\kappa\|_* := \sup_{\pi\in\widehat{G}} \|\widehat{\kappa}(\pi)\|_{op} \qquad \text{(where } \|\cdot\|_{op} \text{ denotes the operator norm)} \ ,$$

it is a Banach algebra which contains $C_c(G)$ and $L^1(G)$ as dense vector subspaces. We also identify it with the collection of μ-measurable fields of uniformly bounded operators.

Throughout this paper, if $\kappa\in\mathcal{S}'(G)$, T_κ denotes the convolution operator

$$T_\kappa : \mathcal{S}(G)\ni f\mapsto f*\kappa \ ,$$

and we keep the same notation for any of its continuous extensions $L^2_b(G)\to L^2_a(G)$ when they exist. With norms possibly infinite, $\|\kappa\|_*$ is equal to the operator norm of $T_\kappa:L^2(G)\to L^2(G)$ by the Plancherel theorem, and is less than $\|\kappa\|_{L^1(G)}$.

For any $a,b\in\mathbb{R}$, it is easy to see that T_κ admits a continuous extension $L^2_b(G)\to L^2_a(G)$ if and only if $(I+\tilde{\mathcal{L}})^{-\frac{b}{\nu}}(I+\mathcal{L})^{\frac{a}{\nu}}\kappa\in C^*(G)$, with equality between the

$L^2_b(G) \to L^2_a(G)$-operator norm and the $C^*(G)$-norm. In this case we may abuse the notation and write

$$\pi(I+\mathcal{L})^{\frac{a}{\nu}}\pi(\kappa)\pi(I+\mathcal{L})^{-\frac{b}{\nu}} \quad \text{instead of} \quad \pi\left((I+\mathcal{L})^{\frac{a}{\nu}}(I+\tilde{\mathcal{L}})^{-\frac{b}{\nu}}\kappa\right) .$$

From the interpolation property of Sobolev spaces (cf Proposition 2.6), we have:

Lemma 2.8. *Let $\kappa \in \mathcal{S}'(G)$ and $a \in \mathbb{R}$. Let $\{\gamma_n,\, n \in \mathbb{Z}\}$ be a sequence of real numbers which tends to $\pm\infty$ as $n \to \pm\infty$. Assume that for any $n \in \mathbb{Z}$, the operator T_κ extends continuously to a bounded operator $L^2_{\gamma_n}(G) \to L^2_{a+\gamma_n}(G)$. Then the operator T_κ extends continuously to a bounded operator $L^2_\gamma(G) \to L^2_{a+\gamma}(G)$ for any $\gamma \in \mathbb{R}$.*

3. Quantisation and symbols classes

As recalled in Introduction, there exists a natural quantisation which is valid on any Lie group of type I. We will present it in this section after defining pre-symbols for which this quantisation makes sense as operator $\mathcal{S}(G) \to \mathcal{S}'(G)$ with G graded Lie groups. Moreover the resulting operators admit integral representations with right convolution kernels and these kernels play a major role in every subsequent proof. We will also define symbol classes and give some examples of symbols.

3.1. **The pre-symbols and their kernels.** A *pre-symbol* is a family of operators $\sigma = \{\sigma(x,\pi),\ x \in G,\ \pi \in \widehat{G}\}$ satisfying:

(1) for each $x \in G$, $\{\sigma(x,\pi),\ \pi \in \widehat{G}\}$ is a μ-measurable field of operators $\mathcal{H}_\pi^\infty \to \mathcal{H}_\pi$,

(2) there exist $\gamma_1, \gamma_2 \in \mathbb{R}$ such that for any $x \in G$,

$$\{\pi(I+\mathcal{L})^{\gamma_1}\sigma(x,\pi)\pi(I+\mathcal{L})^{\gamma_2},\ \pi \in \widehat{G}\} \in C^*(G) \ , \tag{3.1}$$

(3) for any $\pi \in \widehat{G}$ and any $u, v \in \mathcal{H}_\pi$, the scalar function $x \mapsto (\sigma(x,\pi)u, v)_{\mathcal{H}_\pi}$ is smooth over G.

Consequently at each $x \in G$ and $\pi \in \widehat{G}$, the operator $\sigma(x,\pi)$ is densely defined on $\mathcal{H}_\pi$; it is also the case for $X_x^\beta \sigma(x,\pi)$ for any $\beta \in \mathbb{N}^n$.

The second condition implies that for each $x \in G$, the μ-measurable field (3.1) correspond to a distribution $\kappa_{x,\gamma_1,\gamma_2} \in C^*(G)$ which depends smoothly on x; hence σ corresponds to a distribution

$$\kappa(x,\cdot) = \kappa_x := (I+\mathcal{L})^{-\gamma_1}(I+\tilde{\mathcal{L}})^{-\gamma_2}\kappa_{x,\gamma_1,\gamma_2} \ ,$$

which we call its *kernel*. By injectivity of π on $C^*(G)$, $\pi(X_x^\beta \kappa_x) = X_x^\beta \sigma(x,\pi)$.

Examples of pre-symbols are the symbols within the Hörmander classes $S^m_{\rho,\delta}$ defined later on. More concrete examples of pre-symbols which do not depend on $x \in G$ are $\pi(X)^\alpha$, $\alpha \in \mathbb{N}^n$ or the multipliers in $\pi(\mathcal{L})$, that is, $\phi(\pi\mathcal{L})$ with $\phi \in L^\infty(\mathbb{R})$ (for example). Indeed for any $\pi \in \widehat{G}$ the operator $\pi(\mathcal{L})$ is essentially self-adjoint [12] and we denote by E_π its spectral projection, hence giving a meaning to $\phi(\pi\mathcal{L})$. The relation between the spectral projections E and E_π of $\mathcal{L}$ and $\pi(\mathcal{L})$ is

$$\pi(\phi(\mathcal{L})f) = \phi(\pi\mathcal{L})\pi(f) \quad , \quad \phi \in L^\infty(\mathbb{R}) \ , \ f \in L^2(G) \ .$$

It is known [12] that the spectrum of $\pi(\mathcal{L})$ consists of discrete eigenvalues in $(0,\infty)$. This may add a further justification to using the word *quantisation*.

3.2. **The quantisation mapping $\sigma \mapsto Op(\sigma)$.** Our quantisation is analogous to the usual Kohn-Nirenberg quantisation in the Euclidean setting, and has already been noticed by Taylor [17], used indirectly on the Heisenberg group [17, 1] and explicitly on compact Lie groups [14]. It associates to the operator $T = Op(\sigma)$ a pre-symbol σ in the following way (with the same notation as in Subsection 3.1). For any $f \in \mathcal{S}(G)$ and $x \in G$,

$$\int_{\widehat{G}} \mathrm{Tr} \left| \sigma(x,\pi)\widehat{f}(\pi) \right| d\mu(\pi) \leq \sup_{\pi \in \widehat{G}} \| \pi(I+\mathcal{L})^{\gamma_1} \sigma(x,\pi)\ \pi(I+\mathcal{L})^{\gamma_2} \|_{op}$$
$$\int_{\widehat{G}} \mathrm{Tr} \left| \pi\left((I+\mathcal{L})^{-\gamma_1}(I+\tilde{\mathcal{L}})^{-\gamma_2} f \right) \right| d\mu(\pi)\ ,$$

is finite and we can set

$$Tf(x) := \int_{\widehat{G}} \mathrm{Tr}\left(\pi(x)\sigma(x,\pi)\widehat{f}(\pi) \right) d\mu(\pi)\ . \tag{3.2}$$

We have obtained a continuous linear operator $T : \mathcal{S}(G) \to \mathcal{S}'(G)$. By the Schwartz kernel theorem, $T = Op(\sigma)$ has an integral kernel in the distributional sense. However since σ is a pre-symbol, we obtain directly, still in the distributional sense, the following integral representation in terms of the kernel κ defined in Subsection 3.1,

$$Tf(x) = f * \kappa_x(x) = \int_G f(y)\kappa(x, y^{-1}x) dy\ .$$

For example, the pre-symbol σ given by the identity operator on each space $\mathcal{H}_\pi$ is associated with the identity operator on G; its kernel is the Dirac measure at 0 denoted by δ_0 (independent of the point $x \in G$). More generally, for any $\alpha \in \mathbb{N}^n$, the pre-symbol $\pi(X)^\alpha$ is associated with the operator X^α with kernel $(-1)^{|\alpha|}X^\alpha\delta_0$ defined in the sense of distributions via

$$\int_G f(g)(-1)^{|\alpha|}X^\alpha\delta_0(g)dg = \int_G X^\alpha f(g)\delta_0(g)dg = X^\alpha f(0)\ .$$

It is easy to see that the quantisation mapping $\sigma \mapsto T = Op(\sigma)$ is 1-1 and linear. Before defining symbol classes, we need to define difference operators.

3.3. **Difference operators.** Difference operators were defined on compact Lie groups in [14], as acting on Fourier coefficients. Its adaptation to our setting leads us to (densely) define difference operators on $C^*(G)$ viewed as fields. More precisely for any $q \in C(G)$, we set

$$\Delta_q \widehat{f}(\pi) := \widehat{qf}(\pi) = \pi(qf) \quad , \quad f \in C_c(G)\ .$$

This yields a densely defined operator Δ_q on $C^*(G)$ and more generally on $\pi(I+\mathcal{L})^{-\gamma_1}\pi(I+\tilde{\mathcal{L}})^{-\gamma_2}C^*(G)$ for any $\gamma_1, \gamma_2 \in \mathbb{R}$. Note that in general, it is not possible to define an operator Δ_q on each $\mathcal{H}_\pi$; this can be seen quite easily by considering the multiplication by the central variable on the Heisenberg group for example.

The *difference operators* are

$$\Delta^\alpha := (-1)^{|\alpha|}\Delta_{q_\alpha} \quad , \quad \alpha \in \mathbb{N}^n\ ,$$

where the q_α's were defined in Subsection 2.1.1. For technical purposes we insert $(-1)^{|\alpha|}$ in the definition and observe that we have $\Delta^\alpha = \Delta_{q_\alpha^t}$. These operators are densely defined on $C_c(G) \subset C^*(G)$ and satisfy $\Delta^{\alpha_1}\Delta^{\alpha_2} = \Delta^{\alpha_1+\alpha_2}$.

From formula (2.1),

$$\begin{aligned} q_\alpha^t(x)\ f_2 * f_1(x) &= \int_G q_\alpha(x^{-1}y\ y^{-1})\ f_2(y)\ f_1(y^{-1}x)\ dy \\ &= \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2} \int_G q_{\alpha_2}(y^{-1}) f_2(y)\ q_{\alpha_1}(x^{-1}y) f_1(y^{-1}x)\ dy \\ &= \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2}\ (q_{\alpha_2}^t f_2) * (q_{\alpha_1}^t f_1)\ , \end{aligned}$$

we get the *Leibniz formula*:

$$\Delta^\alpha \left(\widehat{f}_1(\pi)\widehat{f}_2(\pi)\right) = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2}\ \Delta^{\alpha_1}\widehat{f}_1(\pi)\ \Delta^{\alpha_2}\widehat{f}_2(\pi)\ . \tag{3.3}$$

The idea of difference operators appear naturally when considering operators on the torus $\mathbb{T}^n$. In this case one recovers forward and backward difference operators on the lattice $\mathbb{Z}^n$. Difference operators were systematically defined and studied on compact Lie groups in [14]. On the Heisenberg group, expressions of a related nature were used to describe the Schwartz space in [9] and with a hypothesis of unitary invariance in [2].

3.4. **The symbol classes $S^m_{\rho,\delta}$.**

Definition 3.1. Let $m, \rho, \delta \in \mathbb{R}$ with $1 \geq \rho \geq \delta \geq 0$ and $\delta \neq 1$. A pre-symbol σ is a *symbol of order m and of type (ρ, δ)* whenever, for each $\alpha, \beta \in \mathbb{N}^n$ and $\gamma \in \mathbb{R}$, the field

$$\{\pi(I+\mathcal{L})^{\frac{\rho[\alpha]-m-\delta[\beta]+\gamma}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi) \pi(I+\mathcal{L})^{-\frac{\gamma}{\nu}}\ ,\ \pi \in \widehat{G}\}\ ,$$

is in $C^*(G)$ uniformly in $x \in G$; this means that we have

$$\sup_{\pi\in\widehat{G},\, x\in G} \|\pi(I+\mathcal{L})^{\frac{\rho[\alpha]-m-\delta[\beta]+\gamma}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi) \pi(I+\mathcal{L})^{-\frac{\gamma}{\nu}}\|_{op} = C_{\alpha,\beta,\gamma} < \infty\ .$$

(The supremum over π is in fact the essential supremum with respect to the Plancherel measure μ.)

The *symbol class* $S^m_{\rho,\delta}$ is the set of symbol of order m and of type (ρ, δ).

We also define $S^{-\infty}_{\rho,\delta} = \cap_{m\in\mathbb{R}} S^m_{\rho,\delta}$ the class of smoothing symbols.

Let us make some comments on this definition:

(1) In the abelian case, that is, $\mathbb{R}^n$ endowed with the addition law and $\mathcal{L}$ being the Laplace operator, $S^m_{\rho,\delta}$ boils down to the usual Hörmander class. In the case of compact Lie groups with $\mathcal{L}$ being the Laplace-Beltrami operator, a similar definition leads to the one considered in [14] since the operator $\pi(I + \mathcal{L})$ is scalar. However here, in the case of non-abelian graded groups, the operator $\mathcal{L}$ can not have a scalar Fourier transform.
(2) The presence of the parameter γ is required to prove that the space of symbols $\cup_{m\in\mathbb{R}} S^m_{\rho,\delta}$ form an algebra of operators later on.
(3) The conditions on α and β are of countable nature and it is also the case for γ. Indeed, by Lemma 2.8, it suffices to prove the property above for one sequence $\{\gamma_n,\ n \in \mathbb{Z}\}$ with $\gamma_n \underset{n\to\pm\infty}{\longrightarrow} \pm\infty$.
(4) A symbol class $S^m_{\rho,\delta}$ is a vector space. And we have the inclusions

$$m_1 \leq m_2 \quad , \quad \delta_1 \leq \delta_2 \quad , \quad \rho_1 \geq \rho_2 \quad \Longrightarrow \quad S^{m_1}_{\rho_1,\delta_1} \subset S^{m_2}_{\rho_2,\delta_2}\ .$$

(5) If $\rho \neq 0$, we will show in Subsections 4.2 and 4.4 that we obtain an algebra of operators with smooth kernels κ_x away from the origin.

If σ is a pre-symbol and $a, b, c \in [0, \infty)$, we set

$$\|\sigma(x,\pi)\|_{S^m_{\rho,\delta},a,b,c} := \sup_{\substack{|\gamma|\leq c \\ [\alpha]\leq a,\, [\beta]\leq b}} \|\pi(I+\mathcal{L})^{\frac{\rho[\alpha]-m-\delta[\beta]+\gamma}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi)\pi(I+\mathcal{L})^{-\frac{\gamma}{\nu}}\|_{op} ,$$

and

$$\|\sigma\|_{S^m_{\rho,\delta},a,b,c} := \sup_{x\in G,\, \pi\in\widehat{G}} \|\sigma(x,\pi)\|_{S^m_{\rho,\delta},a,b,c} .$$

It is a routine exercise to check that for any $m \in \mathbb{R}$, $\rho, \delta \geq 0$, the functions $\|\cdot\|_{S^m_{\rho,\delta},a,b,c}$, $a, b, c \in [0, \infty)$, are semi-norms over the vector space $S^m_{\rho,\delta}$. Furthermore, with Comment 3 above, taking a, b, c as non-negative integers, they endow $S^m_{\rho,\delta}$ of a structure of Fréchet space. The class of smoothing symbols is then equipped with the topology of projective limit.

The pseudo-differential operators of order $m \in \mathbb{R} \cup \{-\infty\}$ and type (ρ, δ) are obtained by quantisation from the symbols of the same order and type, that is,

$$\Psi^m_{\rho,\delta} := Op(S^m_{\rho,\delta}) ,$$

with the quantisation given by (3.2). They inherit a structure of topological vector space from the classes of symbols,

$$\|Op(\sigma)\|_{\Psi^m_{\rho,\delta},a,b,c} := \|\sigma\|_{S^m_{\rho,\delta},a,b,c} .$$

It is not difficult to see from the computations in Subsection 3.2 that any operator $Op(\sigma)$ is a continuous operator $\mathcal{S}(G) \to \mathcal{S}'(G)$; in fact, we can show that T is continuous $\mathcal{S}(G) \to \mathcal{S}(G)$ but the complete proof which uses Theorem 4.4 and Proposition 3.4 will appear elsewhere.

The type $(1, 0)$ is thought of as the basic class of symbols and the types (ρ, δ) as their generalisations, the limitation on the parameters (ρ, δ) coming from reasons similar to the ones in the Euclidean settings. For type $(1, 0)$, we set $S^m := S^m_{1,0}$, $\Psi^m := \Psi^m_{1,0}$ and,

$$\|\sigma(x,\pi)\|_{S^m_{1,0},a,b,c} = \|\sigma(x,\pi)\|_{a,b,c} , \ \|\sigma\|_{S^m_{1,0},a,b,c} = \|\sigma\|_{a,b,c} , \text{ etc} \ldots$$

Before proving that $\cup_{m\in\mathbb{R}} S^m_{\rho,\delta}$ and $\cup_{m\in\mathbb{R}} \Psi^m_{\rho,\delta}$ are stable by composition, let us give some examples.

3.5. **First examples.** As it should be, $\cup_{m\in\mathbb{R}} \Psi^m$ contains the calculus of left invariant differential operators. More precisely the following lemma implies that $\sum_{[\beta]\leq m} c_\beta X^\beta \in \Psi^m$. The coefficients c_α here are constant and it is easy to relax this condition with each function c_α being smooth and bounded as well as all its derivatives.

Lemma 3.2. *For any $\beta_o \in \mathbb{N}^n$, the operator $X^{\beta_o} = Op(\pi(X)^{\beta_0})$ is in $\Psi^{[\beta_o]}$.*

Proof. For any $\alpha \in \mathbb{N}^n$, in the sense of distributions,

$$\int_G f(g)(q^t_\alpha(-1)^{|\beta_o|} X^{\beta_0}\delta_0)(g)dg = \int_G X^{\beta_o}\left\{q^t_\alpha(g)f(g)\right\}\delta_0(g)dg ,$$

is always zero if $[\alpha] < [\beta_o]$ or $[\alpha] = [\beta_o]$ with $\alpha \neq \beta_o$. If $[\alpha] > [\beta_o]$ or $[\alpha] = [\beta_o]$ with $\alpha = \beta_o$, then it is equal to $X^{\beta_o-\alpha} f$ up to some constant $c_{\alpha,\beta_o} \in \mathbb{R}$. Moreover, in

the latter case, we get

$$\begin{aligned}\|f*(q^t_\alpha(-1)^{|\beta_o|}X^{\beta_0}\delta_0)\|_{L^2_{[\alpha]-[\beta_o]+\gamma}} &= |c_{\alpha,\beta_0}|\|X^{\beta_0-\alpha}f\|_{L^2_{[\alpha]-[\beta_o]+\gamma}(G)}\\ &\leq C_{\alpha,\beta_o}\|f\|_{L^2_\gamma(G)}\ .\end{aligned}$$

This shows $\|\pi(I+\mathcal{L})^{\frac{[\alpha]-[\beta_o]+\gamma}{\nu}}\Delta^\alpha\pi(X)^{\beta_o}\pi(I+\mathcal{L})^{-\frac{\gamma}{\nu}}\|_{op}\leq C_{\alpha,\beta_o}$. □

An example of smoothing operator is given by convolution with a Schwartz function:

Lemma 3.3. *If $\kappa\in\mathcal{S}(G)$ then $T_\kappa\in\Psi^{-\infty}$. Furthermore the mapping $\mathcal{S}(G)\ni\kappa\mapsto T_\kappa\in\Psi^{-\infty}$ is continuous.*

Proof. For any $\kappa\in\mathcal{S}(G)$ and $a\geq 0$, we have $(1+\mathcal{L})^a\kappa\in L^1(G)$. Indeed, it is true if $a\in\mathbb{N}$; if $a\notin\mathbb{N}$, then writing

$$(1+\mathcal{L})^a\kappa=\left\{(1+\mathcal{L})^{\lceil a\rceil}\kappa\right\}*G_{a-\lceil a\rceil}\ ,$$

we get

$$\|(1+\mathcal{L})^a\kappa\|_{L^1(G)}\leq\|(1+\mathcal{L})^{\lceil a\rceil}\kappa\|_{L^1(G)}\|G_{a-\lceil a\rceil}\|_{L^1(G)}\ .$$

We have also the same property for $\tilde{\mathcal{L}}$ (either using $\kappa\mapsto\kappa^t$ or adapting the proof above).

Let $m\in\mathbb{R}$. For any $\gamma\in\mathbb{R}$ and $\alpha\in\mathbb{N}^n$ such that γ and $[\alpha]-m+\gamma$ are of the same sign, we have

$$\begin{aligned}&\sup_{\pi\in\widehat{G}}\|\pi(I+\mathcal{L})^{\frac{[\alpha]-m+\gamma}{\nu}}\Delta^\alpha\pi(\kappa)\pi(I+\mathcal{L})^{-\frac{\gamma}{\nu}}\|_{op}\leq\\ &\begin{cases}\|G_\gamma\|_{L^1(G)}\|(1+\mathcal{L})^{\frac{[\alpha]-m+\gamma}{\nu}}q^t_\alpha\kappa\|_{L^1(G)} & \text{if } \gamma,[\alpha]-m+\gamma\geq 0\ ,\\ \|(1+\tilde{\mathcal{L}})^{-\frac{\gamma}{\nu}}q^t_\alpha\kappa\|_{L^1(G)}\|G_{-([\alpha]-m+\gamma)}\|_{L^1(G)} & \text{if } \gamma,[\alpha]-m+\gamma\leq 0\ .\end{cases}\end{aligned}$$

It is now clear that $T_\kappa\in\Psi^m$ and that any semi-norm $\|T\|_{\Psi^m,a,b,c}$ is controlled by some Schwartz semi-norm of κ. □

By Lemma 3.3 and Remark 2.5, if $\phi\in\mathcal{S}(\mathbb{R})$ then $\phi(\mathcal{L})\in\Psi^{-\infty}$. This last consequence could also be obtained via the next example.

The $\mathcal{L}$-multipliers in the following class of functions yields operators in the calculus. We consider the space $\mathcal{M}_m$ of smooth functions ϕ on $[0,\infty)$ satisfying for every $k\in\mathbb{N}$:

$$\|\phi\|_{\mathcal{M}_m,k}:=\sup_{\lambda\geq 0,\,k_1\leq k}\left|(1+\lambda)^{-m+k_1}\partial_\lambda^{k_1}\phi(\lambda)\right|<\infty\ .$$

An important example is $\phi(\lambda)=(1+\lambda)^m$, $m\in\mathbb{R}$.

Proposition 3.4. *Let $m\in\mathbb{R}$ and $\phi\in\mathcal{M}_{\frac{m}{\nu}}$. Then $\phi(\mathcal{L})$ is in Ψ^m and its symbol satisfies*

$$\forall a,b,c\in\mathbb{N}\qquad\exists k\in\mathbb{N},\ C>0\ :\qquad\|\phi(\pi\mathcal{L})\|_{a,b,c}\leq C\|\phi\|_{\mathcal{M}_{\frac{m}{\nu}},k}\ ,$$

with k and C independent of ϕ.

The proof of Proposition 3.4 will be given in a different publication. It is based on Proposition 2.4 and the Cotlar-Stein Lemma.

4. Some properties of symbols, kernels and operators

In this section, we give more explicitly the properties (R1), (R2) and (R4) given in Introduction.

4.1. **First properties of the symbols.** The following properties of the symbol $\sigma \in S^m_{\rho,\delta}$ of an operator with kernel κ_x are not difficult to obtain.

(1) If $\beta_o \in \mathbb{N}^n$ then the symbol $X_x^{\beta_o}\sigma(x,\pi)$ is in $S^{m+\delta[\beta_o]}_{\rho,\delta}$ with kernel $X_x^{\beta_o}\kappa_x$ and,
$$\|X_x^{\beta_o}\sigma(x,\pi)\|_{S^m_{\rho,\delta},a,b,c} \leq C_{b,\beta_o}\|\sigma(x,\pi)\|_{S^m_{\rho,\delta},a,b+[\beta_o],c} \ .$$

(2) If $\alpha_o \in \mathbb{N}^n$ then the symbol $\Delta^{\alpha_o}\sigma(x,\pi)$ is in $S^{m-\rho[\alpha_o]}_{\rho,\delta}$ with kernel $q^t_{\alpha_o}\kappa_x$ and,
$$\|\Delta^{\alpha_o}\sigma(x,\pi)\|_{S^m_{\rho,\delta},a,b,c} \leq C_{b,\beta_o}\|\sigma(x,\pi)\|_{S^m_{\rho,\delta},a+[\alpha_o],b,c} \ .$$

(3) The symbol $\sigma(x,\pi)^*$ is in $S^m_{\rho,\delta}$ with kernel $\kappa_x^* : y \mapsto \bar{\kappa}_x(y^{-1})$ and,
$$\|\sigma(x,\pi)^*\|_{S^m_{\rho,\delta},a,b,c} = \sup_{\substack{|\gamma|\leq c\\ [\alpha]\leq a,\,[\beta]\leq b}} \|\pi(I+\mathcal{L})^{-\frac{\gamma}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi)\pi(I+\mathcal{L})^{\frac{\rho[\alpha]-m-\delta[\beta]+\gamma}{\nu}}\|_{op} \ .$$

(4) Let $\sigma_1 \in S^{m_1}_{\rho,\delta}$ and $\sigma_2 \in S^{m_2}_{\rho,\delta}$ with respective kernels κ_{1x} and κ_{2x}. Then $\sigma(x,\pi) := \sigma_1(x,\pi)\sigma_2(x,\pi)$ defines the symbol σ in $S^m_{\rho,\delta}$, $m = m_1 + m_2$, with kernel $\kappa_{2x} * \kappa_{1x}$; furthermore
$$\|\sigma(x,\pi)\|_{S^m_{\rho,\delta},a,b,c} \leq C\|\sigma_1(x,\pi)\|_{S^{m_1}_{\rho,\delta},a,b,c+\rho a+|m_2|+\delta b}\|\sigma_2(x,\pi)\|_{S^{m_2}_{\rho,\delta},a,b,c} \ .$$
where the constant $C = C_{a,b,c} > 0$ does not depend on σ.

Indeed from the Leibniz rule for Δ^α and X^β, the operator
$$\pi(I+\mathcal{L})^{\frac{[\alpha]-m+\gamma}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi)\pi(I+\mathcal{L})^{-\frac{\gamma}{\nu}} \ ,$$
is a linear combination over $\beta_1, \beta_2, \alpha_1, \alpha_2 \in \mathbb{N}^n$ satisfying $[\beta_1] + [\beta_2] = [\beta]$, $[\alpha_1] + [\alpha_2] = [\alpha]$, of terms
$$\pi(I+\mathcal{L})^{\frac{\rho[\alpha]-m-\delta[\beta]+\gamma}{\nu}} X_x^{\beta_1}\Delta^{\alpha_1}\sigma_1(x,\pi) X_x^{\beta_2}\Delta^{\alpha_2}\sigma_2(x,\pi)\pi(I+\mathcal{L})^{-\frac{\gamma}{\nu}} \ ,$$
whose operator norm is less than
$$\|\pi(I+\mathcal{L})^{\frac{\rho[\alpha]-m-\delta[\beta]+\gamma}{\nu}} X_x^{\beta_1}\Delta^{\alpha_1}\sigma_1(x,\pi)\pi(I+\mathcal{L})^{-\frac{\rho[\alpha_2]-m_2-\delta[\beta_2]+\gamma}{\nu}}\|_{op}$$
$$\|\pi(I+\mathcal{L})^{\frac{\rho[\alpha_2]-m_2-\delta[\beta_2]+\gamma}{\nu}} X_x^{\beta_2}\Delta^{\alpha_2}\sigma_2(x,\pi)\pi(I+\mathcal{L})^{-\frac{\gamma}{\nu}}\|_{op}$$

Consequently, the collection of symbols $\cup_{m\in\mathbb{R}} S^m_{\rho,\delta}$ forms an algebra.

(5) Using the previous point and the left calculus (see Lemma 3.2), if $\sigma \in S^m_{\rho,\delta}$ with kernel κ_x, then $\pi(X)^\beta \sigma\, \pi(X)^{\tilde{\beta}}$ is in $S^{m+[\beta]+[\tilde{\beta}]}$ with kernel $X_y^\beta \tilde{X}_y^{\tilde{\beta}}\kappa_x(y)$.

4.2. **First properties of the kernels.** As expected from pseudo-differential calculi on manifolds such as homogeneous Lie groups, the kernels of the operators of order 0 are of Calderon-Zygmund type in the sense of Coifman-Weiss [4, ch. III]. This claim is a consequence of the following proposition together with the properties of the symbols.

Proposition 4.1. *Assume $\rho \in (0,1]$ and let us fix a homogeneous norm $|\cdot|$ on G. Let $\sigma \in S^m_{\rho,\delta}$ and κ_x the associated kernel.*

Then for each $x \in G$, the distribution κ_x coincides with a smooth function in $G\backslash\{0\}$. Furthermore $(x,y) \mapsto \kappa(x,y)$ is a smooth function on $G \times (G\backslash\{0\})$, and we have the following controls for small and large y.

(1) *There exists $C > 0$ and $a,b,c \in \mathbb{N}$ such that for any $y \in G$ with $|y| < 1$,*
$$|y|^{\frac{Q+m}{\rho}} |\kappa_x(y)| \leq C \sup_{\pi \in \widehat{G}} \|\sigma(x,\pi)\|_{S^m_{\rho,\delta},a,b,c} \ .$$
(2) *For any $M \in \mathbb{N}$, there exists $C > 0$ and $a,b,c \in \mathbb{N}$ such that for any $y \in G$ with $|y| \geq 1$,*
$$|\kappa_x(y)| \leq C \sup_{\pi \in \widehat{G}} \|\sigma(x,\pi)\|_{S^m_{\rho,\delta},a,b,c} |y|^{-M} \ .$$

For example our operators of order 0 have singularities of the type $|y|^{-Q}$ with Q homogeneous dimension strictly greater than the topological dimension. Hence the calculus developed here can not coincide with the Hörmander calculus on $\mathbb{R}^n$ (abelian). This contrasts with the compact case: it was shown in [16] that the calculus developed in [14] on compact Lie groups leads to the usual Hörmander operator classes on $\mathbb{R}^n$ extended to compact connected manifolds.

Before discussing the proof of Proposition 4.1, let us prove the following couple of easy lemmata.

Lemma 4.2. *If $\sigma \in S^m_{\rho,\delta}$ and $a \in \mathbb{R}$ with $m + a > Q/2$, then the distribution $(1+\mathcal{L})^{\frac{a}{\nu}} \kappa_x$ coincides with a square integrable function for each fixed x; furthermore there exists a constant $C = C_{a,m} > 0$, independent of σ, such that we have*
$$\forall x \in G \qquad \|(1+\mathcal{L})^{\frac{a}{\nu}} \kappa_x\|_{L^2(G)} \leq C \sup_{\pi \in \widehat{G}} \|\sigma(x,\pi)\|_{S^m_{\rho,\delta},0,0,0} \ .$$

Proof of Lemma 4.2. By the Plancherel formula and properties of Hilbert-Schmidt operators,
$$\begin{aligned}
\|(1+\mathcal{L})^{\frac{a}{\nu}} \kappa_x\|^2_{L^2(G)} &= \int_{\widehat{G}} \|(1+\mathcal{L})^{\frac{a}{\nu}} \sigma(x,\pi)\|^2_{HS} d\mu(\pi) \\
&\leq \sup_{\pi \in \widehat{G}} \|\pi(I+\mathcal{L})^{\frac{-m}{\nu}} \sigma(x,\pi)\|^2_{op} \int_{\widehat{G}} \|\pi(I+\mathcal{L})^{\frac{m+a}{\nu}}\|^2_{HS} d\mu(\pi) \\
&\leq \sup_{\pi \in \widehat{G}} \|\sigma(x,\pi)\|^2_{S^m_{\rho,\delta},0,0,0} \ \|G_{-(m+a)}\|^2_{L^2(G)} \ ,
\end{aligned}$$
which is finite by the properties of Bessel potentials (see Subsection 2.2.3). □

Lemma 4.3. *If $\sigma \in S^m_{\rho,\delta}$ with $m > Q$, then the associated kernel $\kappa_x(y)$ coincides with a continuous function in y for each $x \in G$. Furthermore there exists a constant $C = C_{a,m} > 0$, independent of σ, such that we have*
$$\forall x, y \in G \qquad |\kappa_x(y)| \leq C \sup_{\pi \in \widehat{G}} \|\sigma(x,\pi)\|_{S^m_{\rho,\delta},0,0,0} \ ,$$

Proof. By Lemma 2.7,
$$\|\kappa_x\|_{L^\infty(G)} \leq C_a \|(1+\mathcal{L})^{\frac{a}{\nu}} \kappa_x\|_{L^2(G)} \quad \text{where} \quad a = \frac{Q+m}{2} > \frac{Q}{2} \ ,$$
and we conclude by Lemma 4.2. □

Consequently, for any $\sigma \in S^m_{\rho,\delta}$ with kernel κ_x, we can apply Lemma 4.3 to the symbol

$$X_x^{\beta_o}\Delta^{\alpha_o}\pi(X)^\beta\sigma\pi(X)^{\tilde\beta} \in S^{m+[\beta]+[\tilde\beta]+\delta[\beta_o]-\rho[\alpha_o]}_{\rho,\delta} \ ;$$

the kernel is given by $\tilde X^{\tilde\beta}X^\beta X_x^{\beta_o}q^t_{\alpha_o}\kappa_x$ (see Subsection 4.1) and, if $m+[\beta]+[\tilde\beta]+\delta[\beta_o]-\rho[\alpha_o]>Q$, it coincides with a continuous and bounded function,

$$\forall x,y\in G \quad |\tilde X_y^{\tilde\beta}X_y^\beta X_x^{\beta_o}q^t_{\alpha_o}\kappa_x(y)| \leq C\sup_{\pi\in\widehat G}\|\sigma(x,\pi)\|_{S^m_{\rho,\delta},a,b,c}$$

with $a=[\alpha_o]$, $b=[\beta_o]$, $c=\rho[\alpha_o]+\delta[\beta_o]+[\tilde\beta]$.

Hence if $\rho>0$ then the kernel of a symbol in $S^m_{\rho,\delta}$ is smooth on $G\times(G\backslash\{0\})$. Furthermore using the homogeneous norm $|\cdot|_p$ defined by (2.2), we have obtained the following weaker version of the first point of Proposition 4.1,

$$|y|_p^{2p}|\kappa_x(y)| \leq C\sup_{\pi\in\widehat G}\|\sigma(x,\pi)\|_{S^m_{\rho,\delta},2p,0,\rho 2p} \tag{4.1}$$

as long as the positive integer p is divisible by $2,\ldots,s$, and such that $m-\rho 2p>Q$. Here $C=C_{m,p}$ is independent of σ.

The estimate (4.1) proves the second point in Proposition 4.1 and is a weaker version of the first. We will not give the proof of this first point because of its length and since this weaker version would be sufficient for the proofs of the other results presented in the rest of the paper. The complete proof will appear elsewhere.

4.3. **A pseudo-differential operator as a limit of 'nice' operators.** The definition of symbols presented above leads to kernels κ_x in the distributional sense and it is often needed to assume that the kernels are 'nice' functions. In this subsection we explain how we proceed to do so.

We fix a non-negative function $\chi_o\in C_c^\infty(\mathbb{R})$ supported in $[1/4,4]$ such that $\chi_o\equiv 1$ on $[1/2,2]$. For any $\epsilon>0$, we write $\chi_\epsilon(x)=\chi_o(\epsilon|x|)$ where the homogeneous norm here is defined by (2.2) with p the smallest positive integer divisible by $2,\ldots,s$.

We denote by $|\pi|$ a 'norm' on $\widehat G$, for example the distance between the co-adjoint orbits of π and 1.

By definition, the $\mathcal{H}_\pi$'s, $\pi\in\widehat G$, form a field of Hilbert spaces for the Plancherel measure. So we can choose a generating sequence of vectors on $\mathcal{H}_\pi$ depending measurably on π. We denote by proj_ϵ the orthogonal projection on the $\lceil\epsilon^{-1}\rceil$-th first vectors of this sequence.

Let $\sigma\in S^m_{\rho,\delta}$. We consider for any $\epsilon\in(0,1)$, the operator

$$\sigma_\epsilon(x,\pi) := \chi_\epsilon(x)1_{|\pi|\leq\epsilon}\sigma(x,\pi)\text{proj}_\epsilon \ .$$

Clearly $\sigma_\epsilon\in S^m_{\rho,\delta}$ and for any $a,b,c\in\mathbb{N}$ there exists $C=C_{m,a,b,c}>0$ such that,

$$\|\sigma_\epsilon\|_{S^m_{\rho,\delta},a,b,c} \leq C\|\sigma\|_{S^m_{\rho,\delta},a,b,c} \ .$$

The corresponding kernel is

$$\kappa_\epsilon(x,y) = \chi_\epsilon(x)\int_{|\pi|\leq\epsilon}\text{Tr}\left(\sigma(x,\pi)\text{proj}_\epsilon\right)d\mu(\pi) \ ,$$

which is smooth in x and y and compactly supported in x. From Proposition 4.1, $\kappa_{\epsilon,x}$ decays rapidly at infinity in y uniformly in x. Furthermore, point-wise for $x\in G$ and $y\in G\backslash\{0\}$, or in the sense of $\mathcal{S}'(G)$-distribution for each $x\in G$, we have the convergence $\kappa_{\epsilon,x}(y)\underset{\epsilon\to 0}{\longrightarrow}\kappa_x(y)$.

Let $T_\epsilon = Op(\sigma_\epsilon)$ be the corresponding operators. For any $f \in \mathcal{S}(G)$, $T_\epsilon f \in C_c^\infty(G)$ and

$$Tf(x) = \lim_{\epsilon\to 0} T_\epsilon f(x) \quad \text{where} \quad T = Op(\sigma) \ .$$

In the proofs of the rest of the paper, we will assume that the kernels of the operators are sufficiently regular and compactly supported in y so that all the performed operations, e.g. composition of operators, convolution of kernels, group Fourier transform of kernels etc... make sense. This can be made rigorous via the procedure described above since we will always obtain controls in $S^m_{\rho,\delta}$-semi-norms.

4.4. **Composition.** We want to prove

Theorem 4.4. *Let $1 \geq \rho \geq \delta \geq 0$ with $\rho \neq 0$ and $\delta \neq 1$. If $T_1 \in \Psi^{m_1}_{\rho,\delta}$ and $T_2 \in \Psi^{m_2}_{\rho,\delta}$ then $T_1T_2 \in \Psi^m_{\rho,\delta}$ with $m = m_1 + m_2$.*

Let us start with some formal considerations. Denoting σ_j and κ_j the symbol and kernel of T_j for $j = 1, 2$, it is not difficult to compute the following expression for the composition $T = T_1T_2$,

$$Tf(x) = \int_G \int_G f(z)\kappa_2(y, z^{-1}y)\kappa_1(x, y^{-1}x)dydz \ ;$$

thus the kernel of T is

$$\kappa_x(w) = \int_G \kappa_2(xz^{-1}, wz^{-1})\kappa_1(x, z)dz \ .$$

Using the Taylor expansion for κ_2 in its first variable, we have

$$\kappa_2(xz^{-1}, \cdot) \approx \sum_\alpha q^t_\alpha(z) X^\alpha_x \kappa_{2x}(\cdot) \quad \text{thus} \quad \kappa_x(w) \approx \sum_\alpha X^\alpha_x \kappa_{2x} * q^t_\alpha \kappa_1(w) \ .$$

Denoting σ the group Fourier transform of κ, we have

$$\sigma(x, \pi) := \pi(\kappa_x) \approx \sum_\alpha \Delta^\alpha \sigma_1(x, \pi) \ X^\alpha_x \sigma_2(x, \pi) \ .$$

From Subsection 4.1, we know that

$$\sum_{[\alpha]\leq M} \Delta^\alpha \sigma_1(x, \pi) \ X^\alpha_x \sigma_2(x, \pi) \in S^{m-(\rho-\delta)M}_{\rho,\delta} \ .$$

Hence the main problem is to control the remainder coming from the use of the Taylor expansion; this is the object of the following lemma.

Lemma 4.5. *We keep the notation defined just above and set*

$$\tau_M := \sigma(x, \pi) - \sum_{[\alpha]\leq M} \Delta^\alpha \sigma_1(x, \pi) \ X^\alpha_x \sigma_2(x, \pi) \ .$$

Let $\beta, \tilde\beta, \beta_o, \alpha_o \in \mathbb{N}^n$. Then there exists $M_o \in \mathbb{N}$ such that for any integer $M > M_o$, there exist $C > 0$ and computable integers $a_1, b_1, c_1, a_2, b_2, c_2$ (independent of σ_1 and σ_2) such that we have

$$\|\pi(X)^\beta \left\{X^{\beta_o}_x \Delta^{\alpha_o} \tau_M\right\} \pi(X)^{\tilde\beta}\|_{op} \leq C\|\sigma_1\|_{S^{m_1}_{\rho,\delta}, a_1, b_1, c_1} \|\sigma_2\|_{S^{m_2}_{\rho,\delta}, a_2, b_2, c_2} \ .$$

Proof of Lemma 4.5. By Subsection 2.1.3, the kernel of τ_M is

$$K_M(x,w) = \int_G \kappa_1(x,z)\ R_{x,M}^{(\kappa_2(\cdot,wz^{-1}))}(z^{-1})dz\ .$$

By Leibniz rules for $X_x^{\beta_o}$ and Δ^{α_o},

$$\begin{aligned}
&X_w^\beta \tilde{X}_w^{\tilde{\beta}} X_x^{\beta_o} \left\{ q_{\alpha_o}^t K_M(x,w) \right\} \\
&= \overline{\sum} \int_G X_x^{\beta_{o1}} q_{\alpha_{o1}}^t(z) \kappa_{1,x}(z)\ X_w^\beta \tilde{X}_w^{\tilde{\beta}} X_x^{\beta_{o2}} R_{x,M}^{(q_{\alpha_{o2}}^t \kappa_2(\cdot,wz^{-1}))}(z^{-1})dz\ ,
\end{aligned}$$

where the sign $\overline{\sum}$ means linear combination, here over $[\alpha_{o1}] + [\alpha_{o2}] = [\alpha_o]$, $[\beta_{o1}] + [\beta_{o2}] = [\beta_o]$, with appropriate constants. Since

$$X_w^\beta \tilde{X}_w^{\tilde{\beta}} X_x^{\beta_{o2}} R_{x,M}^{(q_{\alpha_{o2}}^t \kappa_2(\cdot,wz^{-1}))} = (-1)^{|\beta|} X_{z_1=z}^\beta R_{0,M}^{(X_x^{\beta_{o2}} \tilde{X}_w^{\tilde{\beta}} q_{\alpha_{o2}}^t \kappa_{2,x\cdot}(wz_1^{-1}))}\ ,$$

and $\int Xf\ g = -\int f\ Xg$, the expression $X_w^\beta \tilde{X}_w^{\tilde{\beta}} X_x^{\beta_o} \left\{ q_{\alpha_o}^t K_M(x,w) \right\}$ is equal to

$$\begin{aligned}
&\overline{\sum} \int_G X_z^{\beta_1} X_x^{\beta_{o1}} q_{\alpha_{o1}}^t(z) \kappa_{1,x}(z)\ X_{z_1=z}^{\beta_2} R_{0,M}^{(X_x^{\beta_{o2}} \tilde{X}_w^{\tilde{\beta}} q_{\alpha_{o2}}^t \kappa_{2,x\cdot}(wz^{-1}))}(z_1^{-1})dz \\
&= \overline{\sum} \int_G X_z^{\beta_1} X_x^{\beta_{o1}} q_{\alpha_{o1}}^t(z) \kappa_{1,x}(z)\ R_{0,M-[\beta_2]}^{(X_x^{\beta_{o2}} \tilde{X}_w^{\tilde{\beta}} q_{\alpha_{o2}}^t X_{y=0}^{\beta_2} \kappa_{2,x\cdot y}(wz^{-1}))}(z^{-1})dz\ ,
\end{aligned}$$

where the linear combinations are also over $[\beta_1] + [\beta_2] = [\beta]$. Hence, taking the group Fourier transform, $\pi(X)^\beta \left\{ X_x^{\beta_o} \Delta^{\alpha_o} \tau_M \right\} \pi(X)^{\tilde{\beta}}$ becomes a linear combination over $J = (\beta_{o1}, \beta_{o2}, \alpha_{o1}, \alpha_{o2}, \beta_1, \beta_2)$ of terms

$$E_J := \int_G X_z^{\beta_1} X_x^{\beta_{o1}} q_{\alpha_{o1}}^t(z) \kappa_{1,x}(z) \pi(z)^*\ R_{0,M-[\beta_2]}^{(X_x^{\beta_{o2}} X_{y=0}^{\beta_2} \Delta^{\alpha_{o2}} \sigma_2(x\cdot y) \pi(X^{\tilde{\beta}}))}(z^{-1})dz\ .$$

In each E_J, after $\pi(z)^*$, we insert $I = \pi(I+\mathcal{L})^{-\frac{r}{\nu}} \pi(I+\mathcal{L})^{\frac{r}{\nu}}$ where

$$r := \rho[\alpha_{o2}] - m_2 - \delta(\max_{\alpha \in S_{M-[\beta_2]}} [\alpha] + [\beta_2] + s[\beta_{o2}]) - [\tilde{\beta}]\ ,$$

and S_M defined in (2.3). Let us assume first that $r < 0$. We write

$$-\frac{r}{\nu} = r_o + r' \quad \text{with} \quad r_o = \lceil -\frac{r}{\nu} \rceil \in \mathbb{N} \text{ and } r' \in (-1,0]\ ,$$

and $(I+\mathcal{L})^{r_o} = \overline{\overline{\sum}}_{\beta'} X^{\beta'}$ with $[\beta'] \leq \nu r_o$. We have

$$\begin{aligned}
E_J &= \int_G X_z^{\beta_1} X_x^{\beta_{o1}} q^t_{\alpha_{o1}}(z)\kappa_{1,x}(z)\left(\pi(I+\mathcal{L})^{r_o}\pi(z)\right)^* \\
&\qquad\qquad \pi(I+\mathcal{L})^{r'+\frac{r}{\nu}} R_{0,M-[\beta_2]}^{(X_x^{\beta_{o2}} X_{y=0}^{\beta_2}\Delta^{\alpha_{o2}}\sigma_2(x\cdot y)\pi(X^{\tilde{\beta}}))}(z^{-1})dz \\
&= \overline{\overline{\sum_{[\beta_1']+[\beta_2']\leq \nu r_o}}} \int_G \tilde{X}_z^{\beta_1'}\left\{X_z^{\beta_1} X_x^{\beta_{o1}} q^t_{\alpha_{o1}}(z)\kappa_{1,x}(z)\right\}\ \pi(z)^* \\
&\qquad\qquad \tilde{X}_z^{\beta_2'}\left\{\pi(I+\mathcal{L})^{r'+\frac{r}{\nu}} R_{0,M-[\beta_2]}^{(X_x^{\beta_{o2}} X_{y=0}^{\beta_2}\Delta^{\alpha_{o2}}\sigma_2(x\cdot y)\pi(X^{\tilde{\beta}}))}(z^{-1})\right\} dz \\
&= \overline{\overline{\sum_{[\beta_1']+[\beta_2']\leq \nu r_o}}} \int_G \tilde{X}_z^{\beta_1'} X_z^{\beta_1} X_x^{\beta_{o1}} q^t_{\alpha_{o1}}(z)\kappa_{1,x}(z)\ \pi(z)^* \\
&\qquad\qquad \pi(I+\mathcal{L})^{r'+\frac{r}{\nu}} R_{0,M-[\beta_2+\beta_2']}^{(X_x^{\beta_{o2}} X_{y=0}^{\beta_2+\beta_2'}\Delta^{\alpha_{o2}}\sigma_2(x\cdot y)\pi(X^{\tilde{\beta}}))}(z^{-1})dz\ ,
\end{aligned}$$

therefore,

$$\begin{aligned}
\|E_J\|_{op} &\leq C \sum_{[\beta_1']+[\beta_2']\leq \nu r_o} \int_G \left|\tilde{X}_z^{\beta_1'} X_z^{\beta_1} X_x^{\beta_{o1}} q^t_{\alpha_{o1}}(z)\kappa_{1,x}(z)\right| \\
&\qquad\qquad \|\pi(I+\mathcal{L})^{r'+\frac{r}{\nu}} R_{0,M-[\beta_2+\beta_2']}^{(X_x^{\beta_{o2}} X_{y=0}^{\beta_2+\beta_2'}\Delta^{\alpha_{o2}}\sigma_2(x\cdot y)\pi(X^{\tilde{\beta}}))}(z^{-1})\|_{op} dz\ .
\end{aligned}$$

We fix a homogeneous norm $|\cdot|$ on G. By Proposition 2.2 and Remark 2.3, we have

$$\begin{aligned}
&\|\pi(I+\mathcal{L})^{r'+\frac{r}{\nu}} R_{0,M-[\beta_2+\beta_2']}^{(X_x^{\beta_{o2}} X_{y=0}^{\beta_2+\beta_2'}\Delta^{\alpha_{o2}}\sigma_2(x\cdot y)\pi(X^{\tilde{\beta}}))}(z^{-1})\|_{op} \\
&\leq C \sum_{\alpha\in S_{M-[\beta_2+\beta_2']}} |z|^{[\alpha]} \sup_{|y|\leq b^{M-[\beta_2+\beta_2']+1}|z|} \|\pi(I+\mathcal{L})^{r'+\frac{r}{\nu}} X_y^{\alpha+\beta_2+\beta_2'} X_x^{\beta_{o2}}\Delta^{\alpha_{o2}}\sigma_2(xy)\pi(X^{\tilde{\beta}})\|_{op}\ ,
\end{aligned}$$

while, by Lemmata 2.1 and 3.2, each of the suprema just above can be estimated

$$\begin{aligned}
&\leq C(1+|z|)^{(s-1)s[\beta_{o2}]} \sup_{\substack{x'\in G\\ [\beta"]\leq s[\beta_{o2}]}} \|\pi(I+\mathcal{L})^{r'+\frac{r}{\nu}} X_{x'}^{\alpha+\beta_2+\beta_2'+\beta"}\Delta^{\alpha_{o2}}\sigma_2(x')\pi(I+\mathcal{L})^{\frac{\tilde{\beta}}{\nu}}\|_{op} \\
&\qquad\qquad \leq C(1+|z|)^{(s-1)s[\beta_{o2}]} \sup_{x'\in G} \|\sigma_2(x',\pi)\|_{S^{m_2}_{\rho,\delta},[\alpha_{o2}],[\alpha+\beta_2+\beta_2']+s[\beta_{o2}],[\tilde{\beta}]}\ ;
\end{aligned}$$

this last estimate comes easily from $r'<0$ and $r\leq \rho[\alpha_{o2}]-m_2-\delta[\alpha+\beta_2+\beta_2'+\beta"]$ if $\alpha\in S_{M-[\beta_2+\beta_2']}$.

A similar result holds for $r\geq 0$. Indeed, using $\|\pi(I+\mathcal{L})^{-\frac{r}{\nu}}A\|_{op}\leq \|A\|_{op}$,

$$\begin{aligned}
\|E_J\|_{op} &\leq C_r \int_G \left|X_z^{\beta_1} X_x^{\beta_{o1}} q^t_{\alpha_{o1}}(z)\kappa_{1,x}(z)\right| \\
&\qquad\qquad \|\pi(I+\mathcal{L})^{\frac{r}{\nu}} R_{0,M-[\beta_2]}^{(X_x^{\beta_{o2}} X_{y=0}^{\beta_2}\Delta^{\alpha_{o2}}\sigma_2(x\cdot y)\pi(X^{\tilde{\beta}}))}(z^{-1})\|_{op} dz\ ,
\end{aligned}$$

and we estimate the operator norm as above. This yields

$$\begin{aligned}
&\|\pi(X)^{\beta}\left\{X_x^{\beta_o}\Delta^{\alpha_o}\tau_M\right\}\pi(X)^{\tilde{\beta}}\|_{op} \\
&\qquad \leq C \sup_{x'\in G}\|\sigma_2(x',\pi)\|_{S^{m_2}_{\rho,\delta},a_2,b_2,c_2} \sum I_{\alpha_{o1},\beta_{o1},\beta_1',\beta_1,\beta_{o2}},
\end{aligned}$$

where $(a_2, b_2, c_2) = ([\alpha_o], [\alpha] + [\beta] + s[\beta_o] + \nu(\rho[\alpha_o] + |m_2|), [\tilde{\beta}])$ and,

$$I_{\alpha_{o1},\beta_{o1},\beta_1',\beta_1,\beta_{o2}} := \int_G \left| \tilde{X}_z^{\beta_1'} X_z^{\beta_1} X_x^{\beta_{o1}} q^t_{\alpha_{o1}}(z)\kappa_{1,x}(z) \right| |z|^{[\alpha]}(1+|z|)^{(s-1)s[\beta_{o2}]} dz \ .$$

By Proposition 4.1 and Subsection 4.1, each integral $I_{\alpha_{o1},\beta_{o1},\beta_1',\beta_1,\beta_{o2}}$ converges and is controlled by some semi-norms in σ_1 as long as M is big enough. □

Hence Lemma 4.5 yields the following more precise version of Theorem 4.4.

Corollary 4.6. *Under the hypotheses of Theorem 4.4, writing $T_1 = Op(\sigma_1)$ and $T_2 = Op(\sigma_2)$, there exists a unique symbol $\sigma \in S^m_{\rho,\delta}$ such that $T_1T_2 = Op(\sigma)$. Furthermore*

$$\sigma - \sum_{[\alpha]\leq M} \Delta^\alpha \sigma_1 \ X_x^\alpha \sigma_2 \ \in S^{m-(\rho-\delta)M}_{\rho,\delta} \ ,$$

and the mapping

$$\left\{ \begin{array}{ccc} S^{m_1}_{\rho,\delta} \times S^{m_1}_{\rho,\delta} & \longrightarrow & S^{m-(\rho-\delta)M}_{\rho,\delta} \\ (\sigma_1, \sigma_2) & \longmapsto & \sigma - \sum_{[\alpha]\leq M} \Delta^\alpha \sigma_1 \ X_x^\alpha \sigma_2 \end{array} \right. ,$$

is continuous.

With similar methods, we can prove that $\Psi^m_{\rho,\delta}$ is stable by taking the formal adjoint of an operator, that is, if $T \in \Psi^m_{\rho,\delta}$ then T^* defined via $(Tf_1, f_2)_{L^2} = (f_1, T^*f_2)_{L^2}$ is also in $\Psi^m_{\rho,\delta}$:

Theorem 4.7. *Let $1 \geq \rho \geq \delta \geq 0$ with $\rho \neq 0$ and $\delta \neq 1$. If $T \in \Psi^m_{\rho,\delta}$, then its formal adjoint T^* is also in $\Psi^m_{\rho,\delta}$. More precisely, writing $T = Op(\sigma)$ there exists a unique symbol $\sigma^{(*)} \in S^m_{\rho,\delta}$ such that $T^* = Op(\sigma^{(*)})$. Furthermore*

$$\sigma^{(*)} - \sum_{[\alpha]\leq M} \Delta^\alpha X_x^\alpha \sigma^* \ \in S^{m-(\rho-\delta)M}_{\rho,\delta} \ ,$$

and the mapping

$$\left\{ \begin{array}{ccc} S^m_{\rho,\delta} & \longrightarrow & S^{m-(\rho-\delta)M}_{\rho,\delta} \\ \sigma & \longmapsto & \sigma^{(*)} - \sum_{[\alpha]\leq M} \Delta^\alpha X_x^\alpha \sigma^* \end{array} \right. ,$$

is continuous.

Indeed let us perform formal considerations analogous to the ones for the composition. Let $T = Op(\sigma) \in \Psi^m_{\rho,\delta}$ with kernel κ_x. It is not difficult to compute that the kernel of T^* is $\kappa_x^{(*)}$ given by

$$\kappa_x^{(*)}(y) = \kappa^*_{xy^{-1}}(y) = \bar{\kappa}_{xy^{-1}}(y^{-1}) \ .$$

Using the Taylor expansion for κ_x^* in x, we obtain

$$\kappa_x^{(*)}(y) \approx \sum_\alpha q^t_\alpha(y) X_x^\alpha \kappa_x^*(y) \ .$$

Denoting $\sigma^{(*)}$ the group Fourier transform of $\kappa^{(*)}$, we have

$$\sigma^{(*)}(x,\pi) := \pi(\kappa_x^{(*)}) \approx \sum_\alpha \Delta^\alpha X_x^\alpha \sigma(x,\pi)^* \ .$$

From Subsection 4.1, we know that

$$\sum_{[\alpha]\leq M} \Delta^\alpha X_x^\alpha \sigma(x,\pi)^* \in S_{\rho,\delta}^{m-(\rho-\delta)M} .$$

Hence the main problem is as above to control the remainder coming from the use of the Taylor expansion. The proof proceeds in a similar way and is left to the reader.

Finally, we note that the proof of Theorem 4.4 can be adapted to provide the treatment of the remainder also for composition of operators on compact Lie groups in [14, Theorem 10.7.8].

References

[1] Bahouri, H., Fermanian-Kammerer, C. and Gallagher, I., *Phase-space analysis and pseudo-differential calculus on the Heisenberg group*, Astérisque, **342** (2012). arXiv:0904.4746

[2] Benson, Ch., Jenkins, J. and Ratcliff, G., The spherical transform of a Schwartz function on the Heisenberg group, *J. Funct. Anal.*, **154** (1998), 379–423.

[3] Christ, M., Geller, D., Głowacki, P. and Polin, L., Pseudodifferential operators on groups with dilations. *Duke Math. J.*, **68** (1992), 31–65.

[4] Coifman, R. and Weiss, G., *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Mathematics **242**, Springer-Verlag, Berlin, 1971.

[5] Dixmier, J., *Les C^*-algèbres et leurs représentations*, Deuxième édition. Cahiers Scientifiques, Fasc. XXIX, Gauthier-Villars Éditeur, Paris, 1969.

[6] Dynin, A. S., An algebra of pseudodifferential operators on the Heisenberg groups. Symbolic calculus, *Dokl. Akad. Nauk SSSR*, **227** (1976), 792–795.

[7] Folland, G. B., Subelliptic estimates and function spaces on nilpotent Lie groups, *Ark. Mat.*, **13** (1975), 161–207.

[8] Folland, G. B. and Stein, E., *Hardy spaces on homogeneous groups*, Mathematical Notes **28**, Princeton University Press, 1982.

[9] Geller, D., Fourier analysis on the Heisenberg group. I. Schwartz space, *J. Funct. Anal.*, **36**, (1980), 205–254.

[10] Helffer, B. and Nourrigat, J., Caracterisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie nilpotent gradué, *Comm. Partial Differential Equations*, **4** (1979), 899–958.

[11] Hulanicki, A., A functional calculus for Rockland operators on nilpotent Lie groups, *Studia Math.*, **78** (1984), 253–266.

[12] Hulanicki, A., Jenkins, J. W. and Ludwig, J., Minimum eigenvalues for positive Rockland operators, *Proc. Amer. Math. Soc.*, **94** (1985), 718–720.

[13] Ponge, R., *Heisenberg calculus and spectral theory of hypoelliptic operators on Heisenberg manifolds*, Mem. Amer. Math. Soc., **194** (906), 2008.

[14] Ruzhansky, M. and Turunen, V., *Pseudo-differential operators and symmetries: Background analysis and advanced topics*, Pseudo-Differential Operators: Theory and Applications **2**, Birkhäuser Verlag, 2010.

[15] Ruzhansky, M. and Turunen, V., Global quantization of pseudo-differential operators on compact Lie groups, SU(2) and 3-sphere, *Int Math Res Notices IMRN* (2012), 58 pages, doi: 10.1093/imrn/rns122.

[16] Ruzhansky, M., Turunen, V., and Wirth, J., Hörmander class of pseudo-differential operators on compact Lie groups and global hypoellipticity. arXiv:1004.4396.

[17] Taylor, M. E., *Noncommutative microlocal analysis. I.* Mem. Amer. Math. Soc. **52** (1984). Revised version accessible at http://math.unc.edu/Faculty/met/ncmlms.pdf

[18] Taylor, M. E., *Noncommutative harmonic analysis*, American Mathematical Society, 1986.

Universita degli studi di Padova, DMMMSA, Via Trieste 63, 35121 Padova, Italy
E-mail address: fischer@dmsa.unipd.it

180 Queen's Gate, Department of Mathematics, Imperial College London, London, SW7 2AZ, United Kingdom
E-mail address: m.ruzhansky@imperial.ac.uk